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General Non-extremal Rotating Charged Gödel Black Holes in Minimal Five-Dimensional Gauged Supergravity

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We present the general exact solutions for non-extremal rotating charged black holes in the Gödel universe of five-dimensional minimal supergravity theory. They are uniquely characterized by four non-trivial parameters, namely the mass m, the charge q, the Kerr equal rotation parameter a, and the Gödel parameter j. We calculate the conserved energy, angular momenta and charge for the solutions and show that they completely satisfy the first law of black hole thermodynamics. We also study the symmetry and separability of the Hamilton-Jacobi and the massive Klein-Gordon equations in these Einstein-Maxwell-Chern-Simons-Gödel black hole backgrounds.

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Introduction.—A realistic black hole must localize inside the cosmological background, and a natural consideration for it is our universe (with a possible global rotation). A popular phenomenological theory to describe the universe is given by the standard Friedman-Robertson-Walker (FRW) metric, which represents a rather idealized model of an isotropic homogeneous world filled with perfect fluid. But the standard FRW model is too ideal to describe the global rotation of the universe, since the rotation is a universal phenomenon: all compact objects in the universe rotate.

An exact solution for the rotating universe in four dimensions was found by Gödel [1]. The Gödel universe is an exact solution of Einstein's equation in the presence of a cosmological constant and homogeneous pressureless matter. This space-time solution exhibits several peculiar features including in particular the presence of closed timelike curves (CTCs) through every point.

Recently, the solutions representing the generalization of the Gödel universe to $D = 5$ dimensions, especially in the context of five-dimensional minimal supergravity, have attracted a lot of attention [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. A maximally supersymmetric Gödel-type universe [2] exhibiting most of the peculiar features of the four-dimensional cousin was shown to be an exact solution of minimal supergravity in $D = 5$ dimensions. These solutions are related by T-duality to pp-waves, when uplifted to 10 dimensions [12]. The Gödel-type universes are important due to the possibility of quantizing strings in these backgrounds and their relation to the corresponding limits of super-Yang Mills theories. On the gravitational side, the pp-waves (U-)dual to the Gödel universes arise as the Penrose limits of near-horizon geometries. Quite recently, an exact solution for a stationary Kerr black hole immersed in the rotating Gödel universe has been obtained by Gimon and Hashimoto [4] within the five-dimensional minimal supergravity, and its various properties have been investigated recently in [7, 8, 9, 10, 11]. A procedure was proposed [13, 14] to embed the supersymmetric black ring solutions into the Gödel universe, but no explicit solution was presented.

On the other hand, it is very difficult to find an exact rotating charged solution in higher dimensions. Recently, there has been renewed interest [15, 16, 17, 18, 19, 20, 21, 22] in finding rotating charged solutions of five-dimensional gravity and supergravity. The BMPV black hole [15, 16] is the only asymptotically flat, half supersymmetric, rotating charged black hole with a regular, finite size horizon and finite entropy. Its existence is made possible due to the particular Chern-Simons coupling of minimal $D = 5$ supergravity and the fact that a self-duality condition is allowed to impose on the exterior derivative of the rotation one-form in $D = 5$ dimensions. Later on, further generalizations have been made in [17, 18, 19, 20, 21, 22] to include a nonzero cosmological constant.

As far as we are aware, until recently, an exact solution for a rotating charged black hole localized inside the Gödel universe was not known. A charged extremal black hole with finite horizon area in a Gödel universe was identified in [2, 3], where both the pure Gödel-type universe and the BMPV black hole had also been discussed, but they were presented in two separate solutions, not in a single form. The Kerr-Newman-Gödel black hole [5] and its three-charge's generalization [6] are not exact solutions of $D = 5$ minimal supergravity, but in the extremal limit they indeed yields a remarkable superimposition of the BMPV black hole and the pure Gödel-type universe in one solution, namely the so-called BMPV-Gödel black hole. Therefore, it is an important outstanding problem to find space-times describing a non-extremal rotating charged black hole immersed in the Gödel universe.

In this Letter, we construct an exact solution describing the non-extremal rotating charged black hole localized inside the Gödel universe, and describe its various basic properties. A remarkable feature of the solution is that in the $m = q$ case it superimposes the pure Gödel-type universe and the BMPV black hole in

one solution, similar to that in [5]. Our solution, however, is a faith, charged generalization of the Kerr-Gödel black holes found by Gimon and Hashimotoin [4]. We shall refer to our solution as the Einstein-Maxwell-Chern-Simons-Gödel (EMCS-Gödel) black hole, in order to distinguish it from the one derived in [5]. Just like [4], we also do not require our solutions to preserve any supersymmetry [2]. Another important property is that these EMCS-Gödel black holes still obey the first law (integral and differential Bekenstein-Smarr formulae) of black hole thermodynamics provided the Gödel parameter j is viewed as a thermodynamical variable. In addition, the Hamilton-Jacobi and Klein-Gordon equations are separable in these backgrounds and the space-time admits a reducible Killing tensor.

Metric ansatz.—The bosonic part of the minimal supergravity theory in $D = 5$ dimensions consists of the metric and a one-form gauge field, which are governed by the EMCS equations of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2\Big(F_{\mu\alpha}F_{\nu}^{\ \alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\Big)\,, \qquad (1)$$

$$D_{\nu}\Big(F^{\mu\nu} + \frac{\lambda}{\sqrt{3}\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta\gamma}A_{\alpha}F_{\beta\gamma}\Big) = 0\,. \qquad (2)$$

where $\epsilon^{\mu\nu\alpha\beta\gamma}$ is the Levi-Civita tensor density, and $\lambda = 1$ is the Chern-Simons coupling constant.

To seek an exact solution of the supergravity equations of motion, it is of particular importance to start from a good ansatz for the metric and the gauge potential. A suitable ansatz to our aim assumes the following form

$$\begin{aligned} ds^2 &= -f(r)dt^2 - 2g(r)\sigma_3 dt + h(r)\sigma_3^2 \\ &\quad + \frac{dr^2}{V(r)} + \frac{r^2}{4}d\Omega_3^2\,, \qquad (3) \\ A &= B(r)dt + C(r)\sigma_3\,, \qquad (4) \end{aligned}$$

where the unit 3-sphere $d\Omega_3^2$ and the left invariant form σ_3 are specified by

$$d\Omega_3^2 = d\theta^2 + \sin^2\theta d\psi^2 + \sigma_3^2\,, \quad \sigma_3 = d\phi + \cos\theta d\psi\,. \quad (5)$$

To arrive at the above ansatz for charged generalization of the Kerr-Gödel black hole, we have compared two already-known solutions (a uncharged metric and a charged solution) for the rotating black holes in minimal $D = 5$ supergravity. The first uncharged solution [4] represents the $D = 5$ Kerr black holes embedded in the rotating Gödel universe, its explicit form is given by Eq. (7) below by setting the charge $q = 0$. In this metric, the parameter j defines the scale of the Gödel background and is responsible for the rotation of the universe. The parameter a is related to the rotation of the black hole. When $m = a = 0$, the metric reduces to that of the pure Gödel-type universe [2]. The $D = 5$ Kerr black hole with equal rotation parameters is recovered when $j = 0$.

The second one comes from the charged supergravity solutions in the vacuum backgrounds. The solution for a rotating charged Kerr black hole in five dimensions is again given by Eq. (7) by taking $j = 0$. This solution is the five-dimensional Kerr-Newman black hole with equal-magnitude angular momenta, satisfying the EMCS equations (1) and (2). It is a correct generalization of the four-dimensional Kerr-Newman solution to five dimensions. When $a = 0$, it reduces to the $D = 5$ Reissner-Nordström black hole. The $m = q$ case reproduces the supersymmetric BMPV black hole [15, 16]. This charged solution is also related to the previously known charged solutions [17, 18, 19, 20, 21, 22] in the case without a cosmological constant. In particular, it corresponds to the solution in [18] by taking $\beta = \lambda = 0$ and to that in [19, 20, 21] by $m = p - q$ and $\lambda = 0$. It can also be reduced from the general solution in [22] (with $g = 0$) by setting two angular momenta equal and by redefining the coordinates. The $m = q$ case coincides with the solution in [17] when $g = 0$.

Clearly, our metric ansatz keeps five of the nine isometries of the Gödel universe, generated by ∂_t, and by four generators of the SU(2) $\times$ U(1) subgroup of the SO(4) isometry group acting on S^3 [4]. Our ansatz is also inspired from symmetry and separability of various field equations.

The solutions and basic properties.—We now try to seek an exact charged solution representing the EMCS-Gödel black hole. There are six unknown functions needed to be specified. But among them, a constraint

$$V(r) = 4\frac{g(r)^2 + h(r)f(r)}{r^2} + f(r)\,, \qquad (6)$$

will reduce the actual number to five.

Substituting the ansatz (3) and (4) into Eqs. (1) and (2) will result in a rather complicated set of equations. We are guided by the fact that our solution should reduce to the Kerr-Gödel black hole [4] when the charge parameter vanishes, and to the Kerr-Newman solution when the Gödel parameter is equal to zero. With this in mind, we assume that each unknown function is a polynomial of the radial coordinate, with its coefficients to be determined. Using the GRTensor II program, it is not difficult to check that the following choice (with $\lambda = 1$)

$$\begin{aligned} f(r) &= 1 - \frac{2m}{r^2} + \frac{q^2}{r^4}\,, \\ g(r) &= jr^2 + 3jq + \frac{(2m-q)a}{2r^2} - \frac{q^2 a}{2r^4}\,, \\ h(r) &= -j^2r^2(r^2 + 2m + 6q) + 3jqa \\ &\quad + \frac{(m-q)a^2}{2r^2} - \frac{q^2a^2}{4r^4}\,, \\ V(r) &= 1 - \frac{2m}{r^2} + \frac{8j(m+q)\big[a + 2j(m+2q)\big]}{r^2} \\ &\quad + \frac{2(m-q)a^2 + q^2\big[1 - 16ja - 8j^2(m+3q)\big]}{r^4}\,, \end{aligned}$$

$$B(r) \ = \ \frac{\sqrt{3}q}{2r^2}\,, \quad C(r) = \frac{\sqrt{3}}{2}\Big(jr^2 + 2jq - \frac{qa}{2r^2}\Big)\,, \tag{7}$$

indeed solves the EMCS equations (1) and (2).

The solution presented above is the 'Left' form. Its corresponding 'Right' solution can be generated by the following dual transformations ($\lambda = 1$)

$$\psi \leftrightarrow \phi\,, \quad q \to -q\,, \quad j \to -j\,, \quad \lambda \to -\lambda\,. \tag{8}$$

In the remaining analysis, we shall focus on the 'Left' solution only. The solution is, in general, non-extremal. It is uniquely characterized by four non-trivial parameters, namely the mass m, the charge q, the equal Kerr rotation parameter a, and the Gödel parameter j. When $j = 0$, the metric reduces to the $D = 5$ Kerr-Newman solution. In the case when $q = 0$, we recover the Kerr-Gödel black hole [4]. When the parameter a is set to zero, the solution represents a non-extremal $D = 5$ Reissner-Nordström-Gödel black hole. A supersymmetric Gödel black hole appears when $m = -q$, $a = -2jq$, corresponding to the charged extremal Gödel black hole previously identified in [2, 3]. The $m = q$ case is in particular interesting, it is the BMPV-Gödel black hole, a superimposition of two remarkable solutions in minimal $D = 5$ supergravity, that is, the pure Gödel-type universe and the BMPV black hole. This charged Kerr-Gödel solution with equal rotation parameters is one of the main results in this Letter. In what follows, we shall study its various basic properties.

The space-time has a curvature singularity at $r = 0$, where both the Ricci scalar and the gauge field strength

$$\begin{aligned} R \ &= \frac{1}{3}F_{\mu\nu}F^{\mu\nu} = 16j^2\Big(1 - \frac{m-3q}{r^2} + \frac{6q^2}{r^4}\Big) \\ &- \frac{2q^2\big[1 + 8ja - 8j^2(m+3q)\big]}{r^6} + \frac{4q^2a^2}{r^8}\,, \end{aligned} \tag{9}$$

diverge there.

A salient feature of the solution (7) is that the space-time has an event horizon at r_+ and an inner horizon at r_-, which are determined by $V(r) = 0$, namely, the locations of black hole horizons are

$$\begin{aligned} r_\pm^2 \ &= \ m - 4j(m+q)a - 8j^2(m+q)(m+2q) \pm \sqrt{\delta}\,, \\ \delta \ &= \ \big[m - q - 8j^2(m+q)^2\big]\big[m + q - 2a^2 \\ &\quad -8j(m+2q)a - 8j^2(m+2q)^2\big]\,. \end{aligned} \tag{10}$$

The metric is well behaved at the horizons but the gauge field becomes singular there [11]. Clearly $\delta > 0$ is the condition for the horizon to be well defined. The horizons will degenerate when $\delta = 0$, corresponding to the following different possibilities for j happens,

$$j_\pm = \pm\frac{\sqrt{2(m-q)}}{4(m+q)}\,; \quad \text{or} \quad \frac{-2a \pm \sqrt{2(m+q)}}{4(m+2q)}\,. \tag{11}$$

In these cases, the black hole becomes extremal. On the other hand, a naked singularity will appear when $\delta < 0$.

Two ergospheres appear at $r_{erg}^2 \ = \ m \pm \sqrt{m^2 - q^2}$ where the function $f(r)$ vanishes. They will coincide with each other at $m = \pm q$.

Just as their uncharged counterparts, the EMCS-Gödel solutions can have CTCs. The surface at fixed r where the metric component $g_{\phi\phi} = h(r) + r^2/4$ vanishes is called as the "velocity of light surface" (VLS) or the CTC horizon. The location of CTC horizon $r = r_{\rm VLS}$ is determined by $h(r) + r^2/4 = 0$, namely

$$\begin{aligned} &r^6\big[1 - 4j^2(r^2 + 2m + 6q)\big] + 12jqar^4 \\ &\qquad +2(m-q)a^2r^2 - q^2a^2 = 0\,. \end{aligned} \tag{12}$$

If $r > r_{\rm VLS}$ (when $g_{\phi\phi} < 0$), then ∂_ϕ will be timelike, indicating the presence of CTCs since ϕ is periodic. There will be no CTCs for $r < r_{\rm VLS}$ (when $g_{\phi\phi} > 0$). It should be emphasized that since the Gödel space-time is homogeneous, there is a CTC going through every point in space-time, i.e. the time machine.

The roots of the CTC horizon equation (12) are, in general, very complicated. However, two special cases are relatively simple. In the case of a Reissner-Nordström-Gödel black hole (when $a = 0$), the velocity of light surface is situated at $r_{\rm VLS}^2 = 1/(4j^2) - 2m - 6q$. By contrast, in the case where $m = q = 1/(32j^2)$, the CTC horizons are located at $r_{\rm VLS}^2 = (\sqrt{2} \pm 1)a^{1/2}/(8j^{3/2})$.

It is recognized that for the existence of regular EMCS-Gödel black holes, the four parameters must lie in appropriate ranges so that naked singularities and CTCs are avoided. In other words, they must simultaneously satisfy $\delta > 0$ and $g_{\phi\phi} = h(r) + r^2/4 > 0$. Compared with the uncharged version [4], the parameter space for the charged Kerr-Gödel black hole is much richer than that of a neutral Kerr-Gödel black hole. Therefore it deserves a further analysis in detail, as did in [9].

Thermodynamics of EMCS-Gödel black holes.—For a regular rotating EMCS-Gödel black hole, the horizon topology is a squashed 3-sphere. We now investigate its thermodynamical properties. The Bekenstein-Hawking entropy of the black hole is

$$S = \frac{1}{4}\mathcal{A} = \frac{1}{2}\pi^2 r_+^2\sqrt{4h(r_+) + r_+^2}\,, \tag{13}$$

while the Hawking temperature $T = \kappa/(2\pi)$ is given via the surface gravity

$$\kappa = \frac{r_+V'(r_+)}{2\sqrt{4h(r_+) + r_+^2}} = \frac{r_+^2 - r_-^2}{r_+^2\sqrt{4h(r_+) + r_+^2}}\,. \tag{14}$$

The latter can be obtained by a standard Wick-rotation approach or computed via the standard formula $\kappa^2 = -\frac{1}{2}l_{\mu;\nu}l^{\mu;\nu}|_{r=r_+}$, where the Killing vector $l = \partial_t + \Omega\partial_\phi$ is normal to and becomes null at the horizon.

The angular velocity Ω and the electrostatic potential Φ at the horizon are given by

$$\Omega = \Omega_\phi = g(r_+)/\big[h(r_+) + r_+^2/4\big]\,, \qquad \Omega_\psi = 0\,, \tag{15}$$

$$\Phi = l^\mu A_\mu\big|_{r=r_+} = B(r_+) + C(r_+)\Omega\,. \tag{16}$$

There is a special choice of parameters if they satisfy

$$(m-q)a^2 + 4j(m-q)(m+2q)a - 4j^2(3m+5q)q^2 = 0\,, \tag{17}$$

then $V(r_+) = g(r_+) = 0$. Consequently Ω will vanish at the horizon. This generalizes the non-trivial result ($a = -4jm$) in the case of a Kerr-Gödel black hole [4].

It is remarkable that the conserved energy, angular momenta and charge for the charged Kerr-Gödel black hole,

$$\begin{aligned} M &= \pi\Big[\frac{3}{4}m - j(m+q)a - 2j^2(m+q)(4m+5q)\Big]\,, \\ J &= \frac{1}{2}\pi\Big\{a\big[m - \frac{q}{2} - 2j(m-q)a - 8j^2(m^2+mq \\ &\quad -2q^2)\big] - 3jq^2 + 8j^2(3m+5q)q^2\Big\}\,, \\ Q &= \frac{\sqrt{3}}{2}\pi\big[q - 4j(m+q)a - 8j^2(m+q)q\big]\,, \\ W &= 2\pi\big(m+q\big)\big[a + 2j(m+2q)\big]\,, \end{aligned} \tag{18}$$

satisfy the first law of black hole thermodynamics

$$dM = TdS + \Omega dJ + \Phi dQ + W dj\,, \tag{19}$$

$$\frac{2}{3}M = TS + \Omega J + \frac{2}{3}\Phi Q - \frac{1}{3}Wj\,. \tag{20}$$

To close the integral Bekenstein-Smarr formula, here we have considered the Gödel parameter j as a thermodynamical variable, and introduced its conjugate generalized force W. The conserved charges for M, J, and Q were computed at first by integrating the first law via fixing the parameter j as a constant. Once they are determined, then we can allow j to vary and check the first laws to obtain the expression for W. It is also possible to use $(Wj^2, 1/j)$ as a pair of thermodynamical variables to change the sign of the Gödel work term Wj. Compared with the thermodynamical role [20, 23] played in by the cosmological constant, we can argue that the Gödel parameter behaves just like a cosmological constant in the sense of thermodynamics.

Symmetry and separability of Hamilton-Jacobi and Klein-Gordon equations.—From the inverse metric components and the metric determinant, one can easily infer that not only the Hamilton-Jacobi equation, but also the massive scalar equation with a minimal electro-magnetic coupling term, are capable of separation of variables. This is contrary to the statement made in [10], where the non-separability of Klein-Gordon equation is because the authors had adopted a different coordinate system and expanded the metric in the regime for small j. The separability arises from the fact that the metric ansatz keeps the five isometries of the Gödel universe, generated by ∂_t and four generators of the SU(2) × U(1) group acting on S^3 [4]. The separability also implies that the metric admits a reducible Killing tensor [20]. Details will be published elsewhere.

Discussions.—There are many other interesting issues to explore. The computation of the conserved energy, angular momenta and electric charge of the EMCS-Gödel black holes remains a big challenge, because the naive application of traditional approaches such as the counterterm method fails. At present, there is only one viable work [7] that can do such a job. It remains an open question whether the conserved charges can be computed by other well-known methods. One would like to map out the full parameter spaces of the general non-extremal charged Kerr-Gödel solution. One can lift the $D = 5$ EMCS-Gödel solution to 10 dimensions as a new background for string and M-theory. Ultimately, it would also be interesting to study the causality problem, chronology protection and holography [12]. We hope that the explicit solution describing the non-extremal charged Kerr black holes immersed in the rotating Gödel-type universe will stimulate further insight into these fascinating issues.

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